

# New massive conformal gravity

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## Abstract

We investigate the new massive conformal gravity which is not invariant under conformal transformations, in comparison to the massive conformal gravity. We find five polarization modes of gravitational waves propagating on the Minkowski spacetimes. The stability of Minkowski spacetimes is guaranteed if the mass squared is not negative and the linearized Ricci tensor is employed to describe a massive spin-2 graviton. However, the small Schwarzschild black hole is unstable against the  $s$ -mode massive graviton perturbations.

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# 1 Introduction

Recently, massive conformal gravity (MCG) was proposed as another massive gravity model [1, 2]. This model is composed of a conformally coupled scalar to Einstein-Hilbert term and Weyl-squared term ( $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ ) which are invariant under conformal transformations. The other aspects of the action including the MCG have been studied extensively since eighteen years ago [3, 4, 5, 6, 7, 8].

It was argued that the MCG (2) is a promising quantum gravity model with eight degrees of freedom (DOF) because the conformal symmetry restricts the number of counter-terms arising from the perturbative quantization of the metric tensor [9]. On the contrary, Stelle has shown that the combination of  $R + \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2$  is necessary to improve the perturbative properties of Einstein gravity [10]. This describes also  $8(=2+5+1)$  DOF without scalar. If  $\alpha\beta \neq 0$ , the renormalizability was achieved but the unitarity was violated, showing that the renormalizability is not compatible with the unitarity. Although the Weyl-squared term of providing the massive spin-2 graviton improves the ultraviolet divergence, it induces ghost excitations which spoil the unitarity simultaneously. In this approach, the price one has to pay for making the theory renormalizable is the loss of unitarity. If one excludes the Ricci-squared term, there is no massive spin-0 corrections. Thus, the MCG including the Weyl-squared term solely might not be a candidate for a proper quantum gravity model when one urges to use the metric formalism. Up to now, there is no obvious way to enhance the renormalizability without violating the unitarity in fourth-order gravity.

On the other hand, all undesirable issues of Fierz-Pauli massive gravity with 5 DOF appear when one takes the massless limit of  $m^2 \rightarrow 0$  to match with a massless spin-2 graviton with 2 DOF [11]. Surely, there is a mismatch in DOF: 3 versus 2, which was known to be the van Dam-Veltman-Zakharov (vDVZ) discontinuity [12, 13]. Even though massive gravity has survived as the de Rham-Gabadadze-Tolley (dRGT) gravity [14, 15], the dRGT gravity has still the problems of superluminal propagation and local acausality which mean that it could not be a UV-complete fundamental theory of gravity [16, 17]. Hence, in order to have a better situation, one may propose a new direction of separating the Einstein gravity from the massive gravity: “The Einstein gravity is described by the metric perturbation  $h_{\mu\nu}$  (metric formalism), whereas the massive gravity is described by the linearized Ricci tensor  $\delta R_{\mu\nu}$  (Ricci tensor formalism).” In this view, one does not need to recover the Einstein gravity by taking the massless limit of the massive gravity and, thus

there is no the vDVZ discontinuity.

In this sense, it is worth noting that the MCG might be a candidate for massive gravity model with  $6(=5+1)$  DOF if one expresses a massive spin-2 graviton in terms of the linearized Ricci tensor  $\delta R_{\mu\nu}$  instead of the metric perturbation  $h_{\mu\nu}$ . Here, we are free from linear and non-linear (Boulware-Deser) ghosts because the linearized Einstein equation becomes a second-order differential equation and there is no the vDVZ discontinuity. More importantly, it is well known that the Riemann tensor  $R_{\mu\rho\nu\sigma}$  which causes relative acceleration between test particles is the only measurable field [18]. One uses a null-tetrad basis to compute the Newman-Penrose quantities [19] in terms of the irreducible parts of  $R_{\mu\rho\nu\sigma}$  (the Weyl tensor  $C_{\mu\rho\nu\sigma}$ , the traceless Ricci tensor  $\tilde{R}_{\mu\nu}$ , and the Ricci scalar  $R$ ). According to the analysis in [20], there are six polarization modes of gravitational waves (GWs) in the most general case which will be detected by feasible experiments [21]. For the new massive gravity in three dimensions, one has two polarizations [22]. Also, it would be interesting to see Ref. [23] for cosmologically different observations of the Weyl and Ricci tensors: Ricci-dominated lensing includes large beams of CMB measurements and Weyl-dominated lensing detects narrow beams of SN observations.

The author has shown that the MCG might not be a promising model of massive gravity [2]. The reason is that the non-propagation of the linearized Ricci scalar ( $\delta R = 0$ ) is a strong condition to achieve a massive gravity theory at the linearized level when one uses the Ricci tensor formalism. However, one could not obtain  $\delta R = 0$  because of the conformal symmetry. Adding the Einstein-Hilbert term  $R$  breaks conformal symmetry in the MCG, leading to the new massive conformal gravity (NMCG).

In this work, we wish to test the NMCG as a candidate of massive gravity model with  $6(=5+1)$  DOF. For this purpose, we find five polarization modes of gravitational waves propagating on the Minkowski spacetimes in addition to conformal scalar. We investigate the stability of Minkowski spacetimes as well as the Schwarzschild black hole by using the NMCG. The Minkowski spacetimes is stable against the conformal scalar and linearized Ricci tensor perturbation if the mass squared is not negative ( $m^2 \geq 0$ ). On the other hand, the small Schwarzschild black hole is unstable against the  $s$ -mode massive graviton perturbations under the condition of  $0 < m/\sqrt{2} \leq 1/r_0$  with  $r_0$  the black hole horizon size.

## 2 New massive conformal gravity

We start with the new massive conformal gravity (NMCG) action

$$S_{\text{NMCG}} = \frac{1}{32\pi} \int d^4x \sqrt{-g} \left[ R - \alpha \left( \phi^2 R + 6 \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{2m^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right]. \quad (1)$$

Without the Einstein-Hilbert term, (1) reduces to the massive conformal gravity (MCG)

$$S_{\text{MCG}} = \frac{1}{32\pi} \int d^4x \sqrt{-g} \left[ \alpha \left( \phi^2 R + 6 \partial_\mu \phi \partial^\mu \phi \right) - \frac{1}{m^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right], \quad (2)$$

which is invariant under the full conformal transformations as [1]

$$g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu}, \quad \phi \rightarrow \frac{\phi}{\Omega}. \quad (3)$$

Here  $\Omega(x)$  is an arbitrary function of the spacetime coordinates. Since its conformal transformed action of the Einstein-Weyl gravity has been ruled out by Solar System observations (the deflection of light) as a fourth-order gravity in addition to ghost state problem [6]. This implies that it is meaningless to study the Newtonian approximation to the MCG. Hence, one would be better to consider the MCG as a massive gravity model. In this case, one has a difficulty to obtain the massive graviton equation due to the conformal symmetry, if one does not require a relation of  $\varphi = \delta R/6m^2$  [2].

Adding the Einstein-Hilbert term  $R$  breaks conformal symmetry in the MCG (2), leading to the NMCG (1). Hence, it seems that the idea of imposing exact conformal symmetry as a criterion to choose possible actions of massive gravity does not work one would expects [1]. Furthermore, there is no way to avoid ghost states if one considers any fourth-order gravity models. Both (1) and (2) become unhealthy gravity theories when one uses the metric perturbation. However, the theory becomes a healthy massive gravity theory if one expresses a massive graviton in terms of the linearized Ricci tensor  $\delta R_{\mu\nu}$  instead of metric perturbation  $h_{\mu\nu}$ . This is so because the linearized Einstein equation becomes a second-order differential equation. Similarly, the linearized topologically massive gravity became a first-order theory if one introduces a linearized Einstein tensor  $\delta G_{\mu\nu}$  [24]. In this case, one does not need to introduce a chiral (critical) gravity to avoid the ghost states when one uses the linearized Einstein tensor. In the case of new massive gravity [25], the linearized field equation around the Minkowski vacuum is given by  $(\square - m^2)\delta R_{\mu\nu} = 0$  which is considered as a boosted-up version of the Fierz-Pauli equation  $[(\square - m^2)h_{\mu\nu} = 0]$  [26].

At this stage, we note that even though the last of Weyl-squared term is invariant under conformal transformations (3), we include it as a second-order term because this term provides a unique way of achieving a massive gravity model without a massive spin-0 graviton.

The Einstein equation is derived from (1) as

$$G_{\mu\nu} = \alpha \left[ \phi^2 G_{\mu\nu} + g_{\mu\nu} \nabla^2(\phi^2) - \nabla_\mu \nabla_\nu(\phi^2) + 6\partial_\mu \phi \partial_\nu \phi - 3(\partial\phi)^2 g_{\mu\nu} \right] + \frac{1}{m^2} B_{\mu\nu}, \quad (4)$$

where the Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad (5)$$

and the Bach tensor  $B_{\mu\nu}$  takes the form

$$\begin{aligned} B_{\mu\nu} = & 2 \left( R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) - \frac{2}{3} R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \\ & + \nabla^2 R_{\mu\nu} - \frac{1}{6} \nabla^2 R g_{\mu\nu} - \frac{1}{3} \nabla_\mu \nabla_\nu R. \end{aligned} \quad (6)$$

Its trace is zero ( $B^\mu{}_\mu = 0$ ). On the other hand, the scalar equation is given by

$$\nabla^2 \phi - \frac{1}{6} R \phi = 0. \quad (7)$$

Taking the trace of (4) leads to

$$R = 0 \quad (8)$$

which might be used to simplify the scalar equation (7) as a massless scalar equation

$$\nabla^2 \phi = 0. \quad (9)$$

### 3 Polarization modes of GWs in Minkowski space-times

By setting

$$\bar{R}_{\mu\nu\rho\sigma} = 0, \quad \bar{R}_{\mu\nu} = 0, \quad \bar{R} = 0, \quad \bar{\phi} = \sqrt{\frac{1}{2\alpha}}, \quad (10)$$

we have the Minkowski background as

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(- + ++). \quad (11)$$

In order to develop polarization modes of gravitational waves (GWs), we consider the Ricci tensor and Ricci scalar as first-order functions of the metric perturbation  $h_{\mu\nu}$  in

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (12)$$

Then, we have perturbed Ricci tensor, Ricci scalar, and conformal scalar around the background quantities

$$R_{\mu\nu} = 0 + \delta R_{\mu\nu}, \quad R = 0 + \delta R, \quad \phi = \bar{\phi}(1 + \varphi). \quad (13)$$

We immediately obtain the non-propagation of the Ricci scalar from (8) as

$$\delta R = 0. \quad (14)$$

Considering (14), the perturbed equations are derived from (9) and (4) as

$$\square\varphi = 0, \quad (15)$$

$$\square\delta R_{\mu\nu} - \frac{m^2}{2}(\delta R_{\mu\nu} - 2\partial_\mu\partial_\nu\varphi) = 0. \quad (16)$$

A plane wave solution to (15) is given by

$$\varphi = \varphi_0 e^{iq_\mu x^\mu}, \quad q_\mu q^\mu = 0. \quad (17)$$

On the other hand, a plane solution to (16) takes the form [18]

$$\delta R_{\mu\nu} = A_{\mu\nu} e^{(iqz - \omega t)} + B_{\mu\nu} e^{i(kz - \omega t)} + \text{c.c.}, \quad (18)$$

where

$$A_{\mu\nu} = -2\varphi_0 q_\mu q_\nu, \quad q = \omega, \quad k = \sqrt{\omega^2 - \frac{m^2}{2}} \quad (19)$$

with the frame choice of  $q_\mu = (\omega, 0, 0, q)$  and  $k_\mu = (\omega, 0, 0, k)$  to describe GWs propagating in the  $+z$  direction. So, all the quantities are functions of  $t$  and  $z$  only in this section.

We find all explicit forms of  $\delta R_{\mu\nu}$  as

$$\delta R_{tt} = -2\varphi_0 \omega^2 e^{(iqz - \omega t)} + B_{tt} e^{i(kz - \omega t)} + \text{c.c.}, \quad (20)$$

$$\delta R_{tz} = -2\varphi_0 \omega q e^{(iqz - \omega t)} + B_{tz} e^{i(kz - \omega t)} + \text{c.c.}, \quad (21)$$

$$\delta R_{zz} = -2\varphi_0 q^2 e^{(iqz - \omega t)} + B_{zz} e^{i(kz - \omega t)} + \text{c.c.} \quad (22)$$

and all other components satisfy

$$\delta R_{ij} = B_{ij} e^{i(kz - \omega t)} + \text{c.c.}, \quad (23)$$

where  $i, j = x, y, z$ . Since there is no further constraints on  $B_{\mu\nu}$ , all components of the Riemann tensor in the tetrad basis are given by

$$R_{lklk} = 0, \quad R_{lml\bar{m}} \neq 0, \quad R_{lklm} \neq 0, \quad R_{lkl\bar{m}} \neq 0, \quad (24)$$

where the first term comes from the non-propagation condition of  $\delta R = 0$  (14). They correspond to the Newman-Penrose quantities of Riemann-tensor as [20]

$$\Psi_2 = 0, \quad \Psi_3 \neq 0, \quad \Psi_4 \neq 0, \quad \Phi_{22} \neq 0 \quad (25)$$

whose helicity values are assigned to be  $s = \{0, \pm 1, \pm 2, 0\}$ , respectively when considering Lorentz rotations. These all describe a massive spin-2 graviton with 5 DOF propagating on the Minkowski spacetimes. Together with a conformal scalar  $\varphi$ , the NMCG could describe 6 DOF. If one uses the metric formalism with  $h_{\mu\nu}$ , then the NMCG describes 8 DOF (5 for massive graviton, 1 for conformal scalar, and 2 for massless graviton) [10]. This is another difference in DOF between two formalisms, in addition to the essential difference between second-order and fourth-order linearized equations.

Let us compare the NMCG with the MCG. In the case of the MCG, we had a relation of  $\delta R = 6m^2\varphi$  which implies that  $(\square - m^2)\{\varphi, \delta R\} = 0$ . In this case, one has  $\delta R \neq 0 \rightarrow \Psi_2 \neq 0$ . Hence,  $\Psi_2$  is regarded as a key observational gravitational wave to discriminate between NMCG and MCG. In addition, one has two polarizations of  $\Phi_{12}$  and  $\Phi_{22}$  for the new massive gravity in three dimensions [22].

Finally, we would like to mention the stability of the Minkowski spacetimes. As was shown in (19), we have the dispersion relation for the linearized Ricci tensor propagation

$$\omega^2 = k^2 + \frac{m^2}{2} \quad (26)$$

which implies that  $\omega$  is always real for  $m^2 \geq 0$ , leading to the stability of the Minkowski spacetimes against a massive plane wave (18). However, for the tachyonic mass of  $m^2 = -M^2$ , one may have other dispersion relation

$$\omega^2 = k^2 - \frac{M^2}{2}, \quad (27)$$

which implies the existence of a characteristic wave vector  $k_* = M/\sqrt{2}$  making  $\omega = 0$ . Hence, for  $k > k_*$ , the wave dominates and thus, one achieves the stability (the frequency  $\omega$  is still real and one has oscillations) even though the tachyon mass appears. However, for  $k < k_*$ , the tachyonic mass dominates and thus, the frequency  $\omega$  becomes purely imaginary ( $\omega = -i\Omega$ ). This implies an exponentially growing mode of  $e^{\Omega t}$  which shows an unstable mode. This is an origin of the tachyonic instability in the Minkowski spacetimes which is similar to the Jeans' instability in Newtonian gravity [27]. The Jeans instability is also another pendant for the five-dimensional black string instability [28]. In the next section, however, we will show an instability of  $s$ -mode of the linearized Ricci tensor propagating around the Schwarzschild black hole spacetimes even for  $m^2 > 0$ .

## 4 Instability of massive spin-2 graviton around the Schwarzschild black hole

Considering the background ansatz

$$\bar{R}_{\mu\rho\nu\sigma} \neq 0, \quad \bar{R}_{\mu\nu} = 0, \quad \bar{R} = 0, \quad \bar{\phi} = \sqrt{\frac{1}{2\alpha}}, \quad (28)$$

Eq. (4) and (9) provide the Schwarzschild black hole solution

$$ds_{\text{Sch}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \quad (29)$$

with the metric function

$$f(r) = 1 - \frac{r_0}{r}. \quad (30)$$

It is easy to show that the Schwarzschild black hole (29) is also the solution to the Einstein equation of  $G_{\mu\nu} = 0$  in Einstein gravity. The event horizon appears at  $r = r_0$ .

We introduce the metric and scalar perturbations around the Schwarzschild black hole

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \phi = \bar{\phi}(1 + \varphi) = \sqrt{\frac{1}{2\alpha}}(1 + \varphi). \quad (31)$$

The linearized Einstein equation around the Schwarzschild black hole is given by

$$\begin{aligned} & m^2 \left[ \frac{1}{2} \delta G_{\mu\nu} + \bar{g}_{\mu\nu} \bar{\nabla}^2 \varphi - \bar{\nabla}_\mu \bar{\nabla}_\nu \varphi \right] \\ &= \left[ \bar{\nabla}^2 \delta G_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu} \delta G^{\rho\sigma} \right] - \frac{1}{3} \left[ \bar{\nabla}_\mu \bar{\nabla}_\nu - \bar{g}_{\mu\nu} \bar{\nabla}^2 \right] \delta R, \end{aligned} \quad (32)$$



where the linearized Einstein tensor, Ricci tensor, and Ricci scalar are expressed in terms of  $h_{\mu\nu}$

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\delta R \bar{g}_{\mu\nu}, \quad (33)$$

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\bar{\nabla}^\rho \bar{\nabla}_\mu h_{\nu\rho} + \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\mu\rho} - \bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h\right), \quad (34)$$

$$\delta R = \bar{g}^{\mu\nu} \delta R_{\mu\nu} = \bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h \quad (35)$$

with  $h = h^\rho{}_\rho$ .

Considering (9), its linearized scalar equation is still given by

$$\bar{\nabla}^2 \varphi = 0 \quad (36)$$

whose scalar has a propagating wave being free from unstable modes [29]. Taking the trace of the linearized Einstein equation and using (36), one has

$$-\frac{m^2}{2}\delta R = 0 \quad (37)$$

which implies the non-propagation of linearized Ricci scalar

$$\delta R = 0 \quad (38)$$

for  $m^2 \neq 0$ . We note that  $\delta R = 0$  is confirmed by linearizing  $R = 0$  (8) directly. The choice of  $\delta R = 0$  reflects why we prefer the MCG (2) to the NMCG (1) as a starting action in this work. We stress again that if one does not break conformal symmetry, one could not achieve the non-propagation of the linearized Ricci scalar. Plugging  $\delta R = 0$  and (36) into Eq. (32) leads to the linearized Einstein equation for the linearized Ricci tensor

$$\bar{\nabla}^2 \delta R_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu} \delta R^{\rho\sigma} (= -\Delta_L \delta R_{\mu\nu}) = \frac{m^2}{2} \left[ \delta R_{\mu\nu} - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \varphi \right], \quad (39)$$

where  $\Delta_L$  is the Lichnerowicz operator. Eq. (39) is still difficult to be solved because of coupling  $\delta R_{\mu\nu}$  and  $\varphi$ . In the Minkowski background, Eq. (39) reduces to (16).

Fortunately, Eq. (39) could be expressed compactly by introducing  $\delta \tilde{R}_{\mu\nu} = \delta R_{\mu\nu} - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \varphi$  as

$$\bar{\nabla}^2 \delta \tilde{R}_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu} \delta \tilde{R}^{\rho\sigma} = \frac{m^2}{2} \delta \tilde{R}_{\mu\nu}, \quad (40)$$

where we used an important relation [29]

$$\Delta_L \delta \tilde{R}_{\mu\nu} = \Delta_L \delta R_{\mu\nu}. \quad (41)$$

In proving (41), we have used the relation

$$\Delta_L(\bar{\nabla}_\mu \bar{\nabla}_\nu \varphi) = -\frac{1}{2}(\bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{\nabla}_\nu \bar{\nabla}_\mu) \bar{\nabla}^2 \varphi = 0, \quad (42)$$

where in the second line, we used the linearized scalar equation (36). It is important to note that Eq. (40) could describe the massive spin-2 field (5 DOF) propagating around the Schwarzschild black hole, because  $\delta \tilde{R}_{\mu\nu}$  satisfies the transverse and traceless condition

$$\bar{\nabla}^\mu \delta \tilde{R}_{\mu\nu} = \delta \tilde{R} = 0, \quad (43)$$

where the contracted Bianchi identity was used to prove the transverse condition. After replacing

$$\delta \tilde{R}_{\mu\nu} \rightarrow \delta R_{\mu\nu}, \quad \frac{m^2}{2} \rightarrow m^2, \quad (44)$$

we find the linearized Ricci tensor equation [30]

$$\bar{\nabla}^2 \delta R_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu} \delta R^{\rho\sigma} = m^2 \delta R_{\mu\nu}, \quad (45)$$

where one has found unstable modes of  $e^{\Omega t}$  for

$$0 < m < \frac{\mathcal{O}(1)}{r_0} \quad (46)$$

in fourth-order gravity. The Schwarzschild black hole found from the dRGT gravity is also unstable against the  $s$ -mode of metric perturbations  $h_{\mu\nu}$  [31, 32].

Similarly, we find unstable modes for

$$0 < \frac{m}{\sqrt{2}} < \frac{\mathcal{O}(1)}{r_0} \quad (47)$$

in the NMCG.

In order to find the origin of this instability, we consider a five-dimensional black string described by [28]

$$ds_{\text{BS}}^2 = ds_{\text{Sch}}^2 + dz^2, \quad (48)$$

we have perturbation along an extra direction of the  $z$ -axis

$$h_{AB} = e^{ikz} e^{\hat{\Omega}t} \begin{pmatrix} h_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix}. \quad (49)$$

Using the transverse-traceless gauge condition of  $\bar{\nabla}^\mu h_{\mu\nu} = 0$  and  $h = 0$ , the linearized equation to the Einstein equation of  $R_{AB} = 0$  reduces to

$$\bar{\nabla}^2 h_{\mu\nu} + 2\bar{R}_{\rho\mu\sigma\nu} h^{\rho\sigma} = k^2 h_{\mu\nu}, \quad (50)$$

which describes a massive spin-2 graviton with 5 DOF propagating around the Schwarzschild black hole. One has found a long wavelength perturbation of  $0 < k < k_c \sim \frac{1}{r_0}$  along  $z$ -axis, which gives us an unstable mode of  $e^{\tilde{\omega}t}$ . This is the Gregory-Laframme instability in the black string theory. Comparing (45) with (50), one finds that they are the same by replacing  $\delta R_{\mu\nu}$  and  $m^2$  by  $h_{\mu\nu}$  and  $k^2$ . This implies that the instability of the black hole in the NMCG arises from the massiveness of  $m^2 \neq 0$  where the geometry of extra  $z$  dimension trades for mass [17].

## 5 Discussions

We have studied the new massive conformal gravity as a candidate for massive gravity model with 6 DOF. Using the Ricci tensor formalism, we have found five polarization modes of gravitational waves propagating on the Minkowski spacetimes, in addition to a single conformal scalar.

The stability of Minkowski spacetimes is guaranteed if the mass squared is not negative ( $m^2 \geq 0$ ) and the linearized Ricci tensor was employed to describe a massive spin-2 graviton. However, the small Schwarzschild black hole is unstable against the  $s$ -mode massive graviton perturbations for  $0 < m/\sqrt{2} < 1/r_0$  which corresponds to the positive mass-squared case of  $m^2 > 0$ . This instability is a common feature of the Schwarzschild black hole found from massive gravity theories [31, 32]. Comparing it with the five-dimensional black string instability, the instability of the black hole in the NMCG arises from the massiveness of  $m^2 \neq 0$  where the geometry of extra  $z$  dimension trades for mass.

Consequently, the new massive conformal gravity is regarded as a promising massive gravity model with 6 DOF if one uses the Ricci tensor formalism instead of the metric formalism. The main difference between new massive conformal gravity (1) and massive conformal gravity (2) is that the former has no  $\Psi_2$  gravitational wave due to the non-propagation of Ricci scalar, while the latter has  $\Psi_2$  gravitational wave when one requires a relation between conformal scalar and Ricci scalar ( $\varphi = \delta R/6m^2$ ) additionally. However, the numbers of DOF are the same six for two massive gravity theories.

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